Universal extremal statistics in a freely expanding Jepsen gas

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We study the extremal dynamics emerging in an out-of-equilibrium one-dimensional Jepsen gas of $(N+1)$ hard-point particles. The particles undergo binary elastic collisions, but move ballistically in between collisions. The gas is initially uniformly distributed in a box $[-L, 0]$ with the "leader" (or the rightmost particle) at $X=0$, and a random positive velocity, independently drawn from a distribution $\phi(V)$, is assigned to each particle. The gas expands freely at subsequent times. We compute analytically the distribution of the leader's velocity at time t, and also the mean and the variance of the number of collisions that are undergone by the leader up to time t. We show that in the thermodynamic limit and at fixed time $t \gg 1$ (the so-called "growing regime"), when interactions are strongly manifest, the velocity distribution exhibits universal scaling behavior of only three possible varieties, depending on the tail of $\phi(V)$. The associated scaling functions are entirely different from the usual extreme-value distributions of uncorrelated random variables. In this growing regime the mean and the variance of the number of collisions of the leader up to time t increase logarithmically with t, with universal prefactors that are computed exactly. The implications of our results in the context of biological evolution modeling are pointed out.

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I. INTRODUCTION

The study of the statistics of the maximum or the minimum of a set of random variables, generally referred to as the extreme-value statistics (EVS), is important in diverse areas including disordered systems such as spin-glasses [1] and directed polymers [2–4], turbulent flows [5], sorting and search problems in computer science [6–8], fluctuating interfaces [9–15], granular matter [16], growing networks [17], and models of biological evolution [18–21]. The theory of EVS is simple and well-understood [22,23] when the N random variables V_1, V_2, ..., V_N are statistically *uncorrelated* and each of them is drawn from the same common parent distribution $\phi(V)$. Their maximum

$$V_{\max} = \max(V_1, V_2, \ldots, V_N)$$

is a random variable whose probability density $P(V,N)dV = \mathrm{Prob}[V \leq V_{\max} \leq V+dV, N]$ is known to have a scaling form for large N,

$$P(V,N) \approx \frac{1}{\widetilde{b}_i(N)} G_i\left(\frac{V - \widetilde{a}_i(N)}{\widetilde{b}_i(N)}\right), \quad i = \mathrm{I,II,III}. \quad (1)$$

Here the subscript i refers to the three types of tails of the parent distribution $\phi(V)$, namely:

(I) a tail decaying faster than a power law, such as $\phi(V) \propto V^\beta \exp(-V^\delta)$ when $V \to \infty$, with $\delta > 0$;

(II) a power-law tail, such as $\phi(V) \propto V^{-\beta}$ as $V \to \infty$, where $\beta > 2$; and

(III) a bounded distribution, such as $\phi(V) \propto (V_c - V)^\beta$ when $V \to V_c$, with $\beta > 0$.

The functions $\widetilde{a}_i(N)$ and $\widetilde{b}_i(N)$ are nonuniversal scale factors that depend on the details of $\phi(V)$. However, the scaling functions $G_i(z)$ are universal [22,23], in the sense that they are only of three possible varieties, depending on the three classes I, II, and III, but are otherwise independent of the details of $\phi(V)$. These universal functions are known, respectively, as (I) Fisher-Tippett-Gumbel, (II) Fréchet, and (III) Weibull distributions.

In contrast to the above case of independent identically distributed (i.i.d.) stochastic variables, the EVS is much less understood when there are *correlations* or *interactions* between the random variables. In the presence of static interactions, exact results for the EVS are known only in a few cases, such as for fluctuating $(1+1)$-dimensional interfaces in their steady states [13–15] and for a class of directed polymer problems [2–4]. In this paper, we present exact asymptotic results for the EVS in an interacting particle system, for which the interactions between the random variables are manifest *dynamically*. Our system is a one-dimensional gas consisting of $(N+1)$ identical hard-point particles that undergo binary elastic collisions. At these instantaneous collisions the particles thus merely exchange their velocities, while in between collisions they move freely. Due to the simplicity of the dynamics, this so-called *Jepsen gas* often admits analytical treatments for various externally imposed constraints, and hence has a rather rich history, see, e.g., [24–30]. It has also proved very useful in the study of a class of out-of-equilibrium problems such as the evolution of the "adiabatic piston" [31–34], the Jarzynksi theorem [35], and a quasispecies biological evolution model [19–21]. Moreover, the Jepsen gas also turns out to have important applications in spin transport processes in the one-dimensional nonlinear σ model [36,37].

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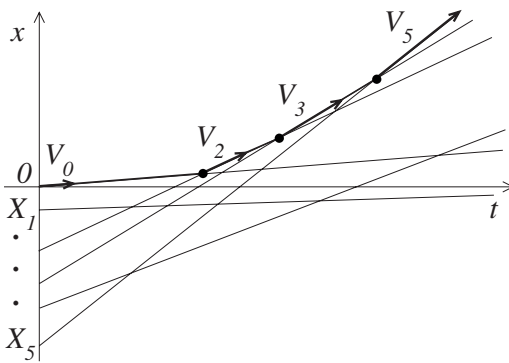†Electronic address: satya.majumdar@u-psud.fr

FIG. 1. A realization of the trajectories for $N=5$ particles plus the leader. The thick line indicates the trajectory of the leader, and the labeled arrows refer to the successive values of its velocity. The points where the leader trajectory gets modified through collisions are indicated by the big, black dots. The thin lines correspond to the trajectories of the other particles.

We start from an initial condition at $t=0$ where the Jepsen gas of $(N+1)$ particles occupies the interval $[-L,0]$ on the real x axis. The extreme-right "zero"-th particle, that we shall conventionally call hereafter the *leader*, is initially located at $X_0=0$. The coordinates X_i, $i=1,\ldots,N$ of the N particles to the left of the leader are uniformly distributed in the interval $-L \leq x < X_0 = 0$. Thus the gas has a uniform initial density $n_0=N/L$. The initial velocities V_i, $i=0,\ldots,N$ of the leader and of the other N particles are independent random variables, identically distributed according to the *parent distribution* $\phi(V)$. For simplicity, we restrict ourselves here to the case of positive velocities, i.e., such that $\phi(V)=0$ for $V<0$. No boundaries affect the dynamics of the system, which is simply a *free expansion*. As the system evolves in time, the particles collide elastically and exchange their velocities. For a given initial condition, the system up to time t is fully described by the set of trajectories $\{X_i+V_i t, i=0,1,\ldots,N\}$. Each of the particles travels along such a trajectory until it collides with another particle, and each collision changes its trajectory. In particular, the velocity of the leader increases whenever it collides with a particle with higher velocity coming from its left, see Fig. 1. For a fixed particle number N, it is obvious that if one waits for a long enough time, then the leader will acquire the largest velocity of the initial set $\{V_i, i=0,1,\ldots,N\}$. Once this happens, the trajectory of the leader remains unchanged for all subsequent times.

In the context of biological evolution of quasispecies, this model was first introduced and studied in Ref. [19]. The population $n_i(t)$ of the ith genotype or species increases exponentially with time, $n_i(t)=n_i(0)\exp(V_i t)$ where the effective rate of reproduction $V_i \geq 0$ defines its "fitness." The logarithmic variable $X_i(t)=\ln[n_i(t)]=X_i(0)+V_i t$ can then be interpreted as the ith trajectory of a Jepsen gas. At $t=0$ the rightmost particle with $X_0=0$ and velocity V_0 is the most fitted genotype; however, if V_0 is not the maximal of the initial velocities, then it will be overtaken successively by better fitted genotypes. At each of these overtaking events the velocity of the leader changes instantaneously by a finite amount, i.e., the fitness of the leading genotype increases discontinuously. These overtaking events thus represent the

so-called punctuation events in the general context of evolution [18]. The important observable here is the number of leading genotype changes up to time t, i.e., the number of collisions that the leader undergoes up to time t. In particular, the total number of punctuation events until the emergence of the eventual "absolute" leader genotype exhibits universal dependencies on the system size N that were investigated numerically in Refs. [19,20] and recently analytically in Ref. [21].

The goal of this paper is to compute analytically two physical quantities of principal interest, namely:

(a) the probability density of the velocity of the leader $P(V,t,N)$ at any time t, and for a fixed number $(N+1)\gg1$ of particles, and

(b) the mean and the variance of the number of collisions undergone by the leader (i.e., the number of leader's trajectory changes) up to a finite time t.

It is clear that in this system, as long as $N\gg1$ is finite, there is a natural time scale $t^*(N)\gg1$ that denotes the time taken by the final trajectory of the leader to emerge. For $t>t^*(N)$, the leader velocity does not change anymore, and as such the leader follows on the trajectory corresponding to the maximum of the initial velocities. Therefore there are obviously two temporal regimes separated by the crossover time scale $t^*(N)$, namely the growing regime and the stationary regime.

The *stationary regime* $[t>t^*(N)\gg1]$. In this regime, the leader's trajectory remains unchanged for all subsequent times. The probability density of the leader velocity $P(V,N,t)$ thus becomes time-independent for $t>t^*(N)\gg1$, and it is given by the probability density $P(V,N)$ of the maximum of the initial velocities. Since the initial velocities are i.i.d., $P(V,N)$ satisfies the scaling form in Eq. (1), where the scaling function $G_i(z)$ has one of the three universal forms (Fisher-Tippett-Gumbel, Fréchet, or Weibull), depending on the tail of $\phi(V)$. Thus one obtains this regime by taking the $t\to\infty$ limit, but keeping $N\gg1$ fixed. The interactions between particles, which are manifest only dynamically, become completely irrelevant in this regime for the velocity distribution of the leader.

The *growing regime* $[1\ll t\ll t^*(N)]$. In this regime the interactions play an important role, and thus the velocity distribution of the leader $P(V,t,N)$ is nontrivial. Since $t\ll t^*(N)$, the finiteness of the system does not affect the leader dynamics; as such, one can study this regime by fixing t and taking the thermodynamic limit, i.e., the $N\to\infty$ limit (at fixed $n_0=N/L$). One can show that the leader velocity distribution approaches an N-independent form $P(V,t)$. In addition, for $t\gg1$ (still in the thermodynamic $N\to\infty$ limit), $P(V,t)$ has a scaling form

$$P(V,t) \approx \frac{1}{b_i(t)} F_i\left(\frac{V-a_i(t)}{b_i(t)}\right), \quad i=\text{I,II,III}, \quad (2)$$

where the index $i=\text{I,II,III}$ refers to the three types of tails of the parent distribution $\phi(V)$ as mentioned before. The functions $a_i(t)$ and $b_i(t)$ are nonuniversal scale factors that depend on the details of $\phi(V)$, but the scaling functions $F_i(z)$ are universal and are of one of three types I, II, and III, but are

otherwise independent of the details of $\phi(V)$. Moreover, the scaling functions $F_i(z)$ are different from the scaling functions $G_i(z)$ in Eq. (1) that characterize the EVS of i.i.d random variables. The exact forms of the scaling functions $F_i(z)$ are detailed in the next section.

We have also computed analytically the mean and the variance of the number of collisions $n_c(t,N)$ undergone by the leader up to time t in a system with $(N+1) \gg 1$ particles. In the stationary regime $t > t^*(N) \gg 1$, it is already known that the statistics of $n_c(t,N)$ becomes time-independent and present universal N-dependence [19–21]. More precisely, $\langle n_c(N) \rangle \approx \xi_i \ln(N)$, $i=\mathrm{I},\mathrm{II},\mathrm{III}$ for large N, where the prefactor ξ_i is universal and its value is specific to each of the three classes $i=\mathrm{I},\mathrm{II},\mathrm{III}$. This result was first conjectured in Ref. [19] and was proved later on analytically in Ref. [21].

In this paper, we compute $\langle n_c(t,N) \rangle$ in the growing regime $[1 \ll t \ll t^*(N)]$ and show that the corresponding mean number of collisions is independent of N for large N, and grows universally with time as $\langle n_c(t) \rangle \approx \gamma_i \ln(n_0 t)$, $i=\mathrm{I},\mathrm{II},\mathrm{III}$, where n_0 is the initial density, and the prefactor γ_i is universal and dependent on the type of tail of $\phi(V)$. Similar universal results are also derived for the variance of $n_c(t,N)$ in the growing regime (see Secs. II and IV for details).

The paper is organized as follows. First, for ready reference, we present in Sec. II a summary of our main results. In Sec. III, we compute the velocity distribution of the leader and provide exact asymptotic results. Section IV discusses the statistics of the number of collisions undergone by the leader up to time t. In Sec. V we provide, for illustration, the full analytical results for a particular choice of $\phi(V)$. Finally, we conclude with a summary and outlook in Sec. VI. To facilitate the reading, most of the lengthy calculations are relegated to Appendixes A–D.

II. SUMMARY OF THE MAIN RESULTS

In this section, we summarize our main results for

(a) the velocity distribution of the leader, and

(b) the mean and the variance of the number of collisions undergone by the leader up to time t.

A. The asymptotic velocity distribution of the leader

As mentioned in the Introduction, the asymptotic velocity distribution of the leader, both in the stationary as well as in the growing regime, is universal, in the sense that it depends only on the tail of the parent distribution $\phi(V)$. As such, three different universality classes emerge. We summarize below these universal behaviors for these three classes.

Class I: Tail decaying faster than a power law as $V \rightarrow \infty$,

$$\phi^{\mathrm{tail}}(V) = \mathcal{A} V^\beta e^{-V^\delta} \tag{3}$$

with $\delta > 0$ and $\mathcal{A}$ a constant related to the details (in particular, the normalization factor) of $\phi(V)$. In this case, the crossover time scales with N as $t^*(N) \propto N$ for large N.

The stationary regime $[t \gg t^*(N) \gg 1]$. The probability density $P(V,N)$ of the leader becomes time-independent and is given by the EVS of the i.i.d. initial velocities. In the limit of
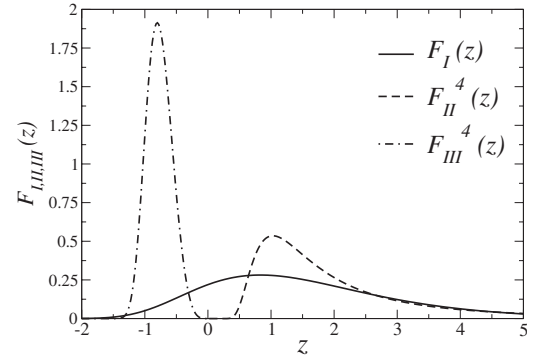

FIG. 2. The universal functions $F_I(z)$ (with continuous line), $F_{II}^4(z)$ (dashed line), and $F_{III}^4(z)$ (dashed-dotted line). The superscript "4" represents the value of the parameter β.

large N (see Sec. IV B for details), $P(V,N)$ satisfies the scaling form in Eq. (1) with

$$\tilde{a}_I(N) \approx \left[\ln\left(\frac{N\mathcal{A}}{\delta} \right) \right]^{1/\delta},$$

$$\tilde{b}_I(N) \approx \frac{1}{\delta} \left[\ln\left(\frac{N\mathcal{A}}{\delta} \right) \right]^{(1-\delta)/\delta}, \tag{4}$$

and the scaling function is equal to the Fisher-Tippett-Gumbel probability density function (PDF),

$$G_I(z) = \exp[-z - \exp(-z)], \quad -\infty < z < \infty. \tag{5}$$

The growing regime $[1 \ll t \ll t^*(N)]$. The velocity distribution $P(V,t)$ of the leader becomes independent of N as $N \rightarrow \infty$ (with t fixed), and has the scaling form described in Eq. (2) with

$$a_I(t) \approx \left[\ln\left(\frac{n_0 t \mathcal{A}}{\delta^2} \right) \right]^{1/\delta},$$

$$b_I(t) \approx \frac{1}{\delta} \left[\ln\left(\frac{n_0 t \mathcal{A}}{\delta^2} \right) \right]^{(1-\delta)/\delta}, \tag{6}$$

and with the universal scaling function

$$F_I(z) = e^{-z} \int_{-\infty}^{z} dU e^{-e^{-U}} = -e^{-z} \operatorname{Ei}(-e^{-z}), \quad -\infty < z < \infty, \tag{7}$$

where $\operatorname{Ei}(z)$ is the exponential-integral function [38]. The profile of $F_I(z)$ is represented in Fig. 2, and its asymptotics are

$$F_I(z) \approx \begin{cases} (z-C)e^{-z} & \text{for } z \rightarrow \infty, \\ \exp(-e^{-z}) & \text{for } z \rightarrow -\infty, \end{cases} \tag{8}$$

where $C=0.577\,215\ldots$ is Euler's constant. Correspondingly, the mean asymptotic velocity of the leader increases with time t as

$$\langle V(t) \rangle \approx [\ln(n_0 t)]^{1/\delta}, \tag{9}$$

and naturally this is also independent on other details of $\phi(V)$.

Class II: Power-law-decaying distributions $\phi(V)$ as $V \to \infty$,

$$\phi^{\text{tail}}(V) = \mathcal{B} V^{-\beta}, \tag{10}$$

where $\beta > 2$ in order to ensure the finiteness of the first moment of $\phi(V)$, and $\mathcal{B}$ is a constant. In this case the crossover time $t^*(N) \approx N^{(\beta-2)/(\beta-1)}$ for large N, and the leader velocity distribution has the following asymptotic universal form in the two regimes.

Stationary regime. The time-independent $P(V,N)$ satisfies the scaling form in Eq. (1), with

$$\tilde{a}_{II}(N) = 0,$$

$$\tilde{b}_{II}(N) \approx \left(\frac{\mathcal{B} N}{\beta - 1} \right)^{1/(\beta-1)}, \tag{11}$$

and the scaling function is the Fréchet PDF,

$$G_{II}(z) = \begin{cases} \dfrac{(\beta - 1)}{z^\beta} \exp\left(-\dfrac{1}{z^{\beta-1}} \right) & \text{for } z \geq 0, \\ 0 & \text{for } z < 0. \end{cases} \tag{12}$$

Growing regime. The leader velocity distribution $P(V,t)$ becomes independent of N as $N \to \infty$, and satisfies Eq. (2) with

$$a_{II}(t) = 0, \quad b_{II}(t) \approx \left[\frac{n_0 t \mathcal{B}}{(\beta - 1)(\beta - 2)} \right]^{1/(\beta-2)}. \tag{13}$$

The universal scaling function $F_{II}(z)$ depends only on the parameter β [but is otherwise independent of the details of $\phi(V)$]:

$$F_{II}^\beta(z) = (\beta - 1) z^{-\beta} \int_{z^{2-\beta}}^\infty dU \, U^{-(\beta-1)/(\beta-2)} e^{-U}, \tag{14}$$

that can also be written as

$$F_{II}^\beta(z) = (\beta - 1)(\beta - 2) z^{-\beta} \int_0^z dU \, e^{-U^{2-\beta}}, \tag{15}$$

and is defined for $z \in [0, \infty)$. For $z < 0$, $F_{II}^\beta(z) = 0$ trivially. Its limiting behavior is given by

$$F_{II}^\beta(z) \approx \begin{cases} \dfrac{(\beta - 1)(\beta - 2)}{z^{\beta-1}} & \text{for } z \to \infty, \\ \dfrac{\beta - 1}{z} \exp(-z^{2-\beta}) & \text{for } z \to 0^+. \end{cases} \tag{16}$$

The mean velocity of the leader increases asymptotically as a power law,

$$\langle V(t) \rangle \approx \frac{\beta - 1}{\beta - 2} \Gamma\left(\frac{\beta - 3}{\beta - 2} \right) b_{II}(t)$$

$$\approx \frac{\beta - 1}{\beta - 2} \left[\frac{\mathcal{B}}{(\beta - 1)(\beta - 2)} \right]^{1/(\beta-2)} \Gamma\left(\frac{\beta - 3}{\beta - 2} \right)$$

$$\times (n_0 t)^{1/(\beta-2)}. \tag{17}$$

The above expression is finite provided that $\beta > 3$. Thus although $P(V,t)$ exists for all $\beta > 2$, its first moment is infinite for $2 < \beta \leq 3$.

Class III: Distribution $\phi(V)$ with a finite maximum velocity V_c,

$$\phi^{\text{tail}}(V) = \mathcal{C}(V_c - V)^\beta \quad \text{for } V \lesssim V_c, \tag{18}$$

with $\beta > 0$. In this case, $t^*(N) \approx N^{(\beta+2)/(\beta+1)}$ for large N.

Stationary regime. $P(V,N)$ satisfies the scaling form of Eq. (1) with

$$\tilde{a}_{III}(N) = V_c,$$

$$\tilde{b}_{III}(N) \approx \left(\frac{\mathcal{C} N}{\beta + 1} \right)^{-1/(\beta+1)}, \tag{19}$$

and the scaling function is the Weibull PDF,

$$G_{III}(z) = \begin{cases} 0 & \text{for } z \geq 0, \\ (\beta + 1)|z|^\beta \exp[-|z|^{\beta+1}] & \text{for } z < 0. \end{cases} \tag{20}$$

Growing regime. In this regime, the leader velocity distribution $P(V,t)$ becomes independent of N as $N \to \infty$, and satisfies to Eq. (2) with

$$a_{III}(t) = V_c, \quad b_{III}(t) = \left[\frac{(\beta + 1)(\beta + 2)}{n_0 t \mathcal{C}} \right]^{1/(\beta+2)}. \tag{21}$$

The β-dependent universal scaling function is given by

$$F_{III}^\beta(z) = (\beta + 1)(\beta + 2)|z|^\beta \int_{|z|}^\infty dU \, e^{-U^{\beta+2}} \tag{22}$$

for $z \in (-\infty, 0]$, and $F_{III}^\beta(z) = 0$ for $z > 0$. Its limiting behavior is given by

$$F_{III}^\beta(z) \approx \begin{cases} \dfrac{(\beta + 1)}{|z|} \exp[-|z|^{\beta+2}] & \text{for } z \to -\infty, \\ (\beta + 1)\Gamma\left(\dfrac{1}{\beta + 2} \right)|z|^\beta & \text{for } z \to 0^-. \end{cases} \tag{23}$$

The asymptotic mean velocity of the leader approaches the maximum allowed value V_c as a power law,

$$\langle V(t) \rangle \approx V_c - \frac{\beta + 1}{\beta + 2} \Gamma\left(\frac{\beta + 3}{\beta + 2} \right) b_{III}(t)$$

$$\approx V_c - \frac{\beta + 1}{\beta + 2} \left[\frac{(\beta + 1)(\beta + 2)}{\mathcal{C}} \right]^{1/(\beta+2)}$$

$$\times \Gamma\left(\frac{\beta + 3}{\beta + 2} \right) (n_0 t)^{-1/(\beta+2)}. \tag{24}$$

B. Collision statistics of the leader

We have also computed analytically the mean and the variance of $n_c(t,N)$, the number of collisions undergone by the leader until a given time t and for a given $N \gg 1$. As before, one is led to consider the two regimes, namely the stationary regime $[t > t^*(N)]$ and the growing regime $[1 \ll t \ll t^*(N)]$.

In the *stationary regime* the mean and the variance become time-independent and they increase logarithmically with N for large N. The mean behaves as

$$\langle n_c(N) \rangle \approx \xi_i \ln(N), \quad i = \mathrm{I,II,III}, \tag{25}$$

where the universal prefactor ξ_i depends on the three classes of $\phi(V)$,

$$\xi_I = \frac{1}{2},$$

$$\xi_{II} = \frac{\beta - 2}{2\beta - 3}, \tag{26}$$

and

$$\xi_{III} = \frac{\beta + 2}{2\beta + 3}.$$

Similarly, the variance in the stationary regime, for large N, behaves as

$$\langle n_c^2(N) \rangle - \langle n_c(N) \rangle^2 \approx \sigma_i \ln(N), \quad i = \mathrm{I,II,III}, \tag{27}$$

where

$$\sigma_I = \frac{1}{4},$$

$$\sigma_{II} = \frac{(\beta - 2)(2\beta^2 - 6\beta + 5)}{(2\beta - 3)^3}, \tag{28}$$

and

$$\sigma_{III} = \frac{(\beta + 2)(2\beta^2 + 6\beta + 5)}{(2\beta + 3)^3}.$$

The above results were first derived analytically in Ref. [21] using a different method. This paper thus provides an alternative derivation.

In the *growing regime*, we show that the mean number of collisions increases logarithmically with t,

$$\langle n_c(t) \rangle \approx \gamma_i \ln(n_0 t), \quad i = \mathrm{I,II,III}. \tag{29}$$

The universal prefactor γ_i is characteristic to each of the classes of the parent-distribution $\phi(V)$, namely

$$\gamma_I = \frac{1}{2},$$

$$\gamma_{II} = \frac{\beta - 1}{2\beta - 3}, \tag{30}$$

and

$$\gamma_{III} = \frac{\beta + 1}{2\beta + 3}.$$

Moreover, as discussed in Sec. IV, one can also infer the variance of the number of collisions,

$$\langle n_c^2(t) \rangle - \langle n_c(t) \rangle^2 \approx \eta_i \ln(n_0 t), \quad i = \mathrm{I,II,III}, \tag{31}$$

where

$$\eta_I = \frac{1}{4},$$

$$\eta_{II} = \frac{(\beta - 1)(2\beta^2 - 6\beta + 5)}{(2\beta - 3)^3},$$

and

$$\eta_{III} = \frac{(\beta + 1)(2\beta^2 + 6\beta + 5)}{(2\beta + 3)^3}. \tag{32}$$

The fact that $\langle n_c^2(t) \rangle - \langle n_c(t) \rangle^2 \neq \langle n_c(t) \rangle$ indicates, contrary to previous claims [33], that the collision process in the thermodynamic limit of the Jepsen gas is *not Poissonian*.

III. VELOCITY DISTRIBUTION FUNCTION OF THE LEADER

A. General relations

In order to establish the characteristics of the leader's stochastic dynamics, we shall follow the type of reasoning and the convenient notations of Ref. [32]. Recall that the particles simply exchange velocities upon collisions and cannot move across each other; therefore at any time t the leader rides the instantaneous *rightmost trajectory* among the free trajectories $\{X_i + V_i t, i = 0, 1, \ldots, N\}$. As the number of particles on the left-hand side of the leader is a conserved quantity, we can identify this instantaneous trajectory $(X_p + V_p t)$ of the leader by imposing

$$\sum_{i=0, i \neq p}^{N} \theta(X_p + V_p t - X_i - V_i t) = N, \tag{33}$$

θ being the Heaviside step function.

These elements are sufficient to find the conditional probability distribution of the leader $P_L(X, V, t | V_0)$ at time $t > 0$ as

$$P_L(X, V, t | V_0) = \left\langle \sum_{p=0}^{N} \delta(X - X_p - V_p t) \delta(V - V_p) \right.$$
$$\left. \times \delta_{Kr}\left(N, \sum_{i=0, i \neq p}^{N} \theta(X_p + V_p t - X_i - V_i t)\right) \right\rangle, \tag{34}$$

where the brackets $\langle \cdots \rangle$ denote averaging over the initial positions and velocities of the gas particles in $[-L, 0]$. The initial condition is obviously

$$P_L(X, V, t = 0 | V_0) = \delta(X) \delta(V - V_0). \tag{35}$$

As shown in Appendix A, one finds

$$P_L(X,V,t|V_0) = \oint_\Gamma \frac{dz}{2\pi i z} [A(z,X/t|L/t)]^N$$
$$\times \left\{ \delta(X-Vt)\,\delta(V-V_0) \right.$$
$$+ n_0\phi(V)\,\theta(Vt-X)\,\theta(X-Vt+L)$$
$$\left. \times \frac{1+(z-1)\theta(V_0 t-X)}{A(z,X/t|L/t)} \right\}. \qquad (36)$$

Here Γ is the unit-circle in the complex z-plane, and

$$A(z,X/t|L/t) = 1 + (z-1)\left[\int_{(L+X)/t}^\infty dU\,\phi(U) \right.$$
$$\left. + \frac{t}{L}\int_{X/t}^{(L+X)/t} dU\left(U-\frac{X}{t}\right)\phi(U) \right], \qquad (37)$$

and $n_0 = N/L$ is the initial density. One can also check the normalization of $P_L(X,V,t|V_0)$, see Appendix A.

We now focus on the growing regime, i.e., we fix the time t and take the *thermodynamic limit* $L \to \infty$, $N \to \infty$ at constant n_0. One finds then for the conditional probability distribution of the leader

$$P(X,V,t|V_0) = \oint_\Gamma \frac{dz}{2\pi i z}$$
$$\times \exp\left[n_0 t(z-1)\int_{X/t}^\infty dU\left(U-\frac{X}{t}\right)\phi(U) \right]$$
$$\times \{ \delta(X-V_0 t)\,\delta(V-V_0) + n_0\phi(V)\,\theta(Vt-X)$$
$$\times [1+(z-1)\theta(V_0 t-X)] \}. \qquad (38)$$

Introducing the function

$$\alpha(W) \equiv \int_W^\infty dU(U-W)\phi(U), \qquad (39)$$

one finally obtains:

$$P(X,V,t|V_0) = e^{-n_0 t\alpha(V_0)}\delta(X-V_0 t)\,\delta(V-V_0)$$
$$+ n_0\phi(V)e^{-n_0 t\alpha(X/t)}\theta(Vt-X)\,\theta(X-V_0 t). \qquad (40)$$

This result has a simple physical interpretation, based on a "flux argument." Let V be the instantaneous velocity of the leader at time t. The leader's trajectory can get bypassed only by the trajectories with higher initial velocities $U > V$; then the rate at which the trajectory of the leader gets bypassed is proportional to the flux of particle trajectories of slopes higher than V. This flux is clearly proportional to the particle density n_0 and to the relative velocity $(U-V)\theta(U-V)$. Therefore $n_0\alpha(V) = n_0\int_V^\infty dU(U-V)\phi(U)$ represents the total instantaneous rate at which the leader's trajectory with velocity V gets bypassed by other trajectories.

The first term on the right-hand side (rhs) of Eq. (40) represents the probability that the leader's trajectory has never been bypassed until time t. One notices that this probability is exponentially decreasing with time. In this case, the
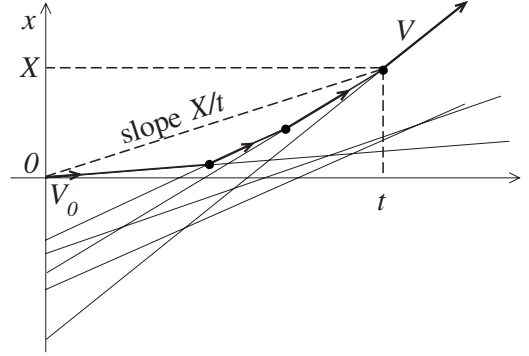

FIG. 3. Illustration of the "flux argument" in deducing the conditional probability distribution function $P(X,V,t|V_0)$ of the leader in the thermodynamic limit (see the main text).

final velocity is V_0 and the final position is $V_0 t$, thus explaining the two δ factors. Indeed, as seen above, in an infinitesimal time dt the trajectory of the leader with velocity V_0 gets hit with probability $n_0\alpha(V_0)dt$, and so it does not get hit with probability $1-n_0\alpha(V_0)dt$. As such, the probability for the leader to keep its initial trajectory until time t is $[1-n_0\alpha(V_0)dt]^{t/dt} \to \exp[-n_0 t\alpha(V_0)]$.

Let us turn now to the second term of the rhs of Eq. (40), which takes into account all the situations when the velocity of the leader got modified through collisions. The leader is at X at time t, with velocity V. For this event to happen, no trajectory must hit the straight line of slope X/t until time t, see Fig. 3. All possible trajectories of the leader must lie below this line, and (according to the argument above) this happens with probability $\exp[-n_0 t\alpha(X/t)]$. Now, out of all the trajectories satisfying this criterion, we are interested only in those that actually hit the line of slope X/t exactly at time t, and exactly with velocity V; this fraction is $n_0\phi(V)$. Thus the total probability for the leader to be at X with velocity V is $n_0\phi(V)\exp[-n_0 t\alpha(X/t)]$ (the two θ functions are obvious).

The conditional coordinate distribution for the leader is obtained by integrating $P(X,V,t|V_0)$ over V:

$$P(X,t|V_0) = e^{-n_0 t\alpha(V_0)}\delta(X-V_0 t) + n_0 e^{-n_0 t\alpha(X/t)}$$
$$\times \int_{X/t}^\infty dV\,\phi(V)\,\theta(X-V_0 t)$$
$$= \frac{\partial}{\partial X}\{ e^{-n_0 t\alpha(X/t)}\theta(X-V_0 t) \}, \qquad (41)$$

and a detailed discussion of its long-time properties, corresponding to the diffusive regime for the particle, can be found in Ref. [27].

We now consider exclusively the stochastic behavior of the velocity of the leader. The *conditional* velocity distribution is obtained by integrating $P(X,V,t|V_0)$ over X:

$$P(V,t|V_0) = e^{-n_0 t\alpha(V_0)}\delta(V-V_0) + n_0 t\phi(V)$$
$$\times \int_{V_0}^V dW\,e^{-n_0 t\alpha(W)}\theta(V-V_0). \qquad (42)$$

Moreover, by averaging over the initial velocity V_0 of the

leader, one obtains the *velocity distribution*:

$$P(V,t) = \int_0^\infty dV_0 \phi(V_0) P(V,t|V_0)$$

$$= \phi(V) e^{-n_0 t \alpha(V)} + n_0 t \phi(V) \int_0^V dV_0 \phi(V_0)$$

$$\times \int_{V_0}^V dW e^{-n_0 t \alpha(W)}. \tag{43}$$

Based on the properties of the function $\alpha(W)$ (see Appendix B), one finally reaches an expression that proves to be more convenient for further analysis,

$$P(V,t) = \phi(V) e^{-n_0 t \mu} + n_0 t \phi(V) \int_0^V dW e^{-n_0 t \alpha(W)}. \tag{44}$$

Here $\mu > 0$ is the first moment of $\phi(V)$,

$$\mu = \int_0^\infty dV V \phi(V). \tag{45}$$

B. Long-time behavior of $P(V,t)$

Let us inspect Eq. (44) in the limit $t \gg (n_0 \mu)^{-1}$. The first term on the rhs is exponentially decreasing in time, so it can be neglected in this long-time limit. In the second term, due to the presence of the exponential $\exp[-n_0 t \alpha(W)]$, the main contribution to the integral will come from the W sector for which $\alpha(W)$ is small; in view of the property (i) in Appendix B, this happens for large values of W, so one can write with a good approximation

$$P(V,t) \approx n_0 t \phi(V) \int_0^\infty dW \exp[-n_0 t \alpha^{as}(W)]. \tag{46}$$

Here "as" designates the asymptotic, large-W behavior of $\alpha(W)$, which, in view of the definition (39), is determined by the asymptotic behavior of $\phi(V)$. We are thus led to consider the tail of the distribution $\phi(V)$, according to the three classes discussed in Sec. II.

Class I, Eq. (3). In this case one finds

$$\alpha^{as}(W) \approx \frac{\mathcal{A}}{\delta^2} W^{\beta+2(1-\delta)} \exp(-W^\delta). \tag{47}$$

As shown in Appendix C, this leads to the scaling form (2) with the scaling function F_I in Eq. (7) and the scaling parameters (6).

Class II, Eq. (10). One has

$$\alpha^{as}(W) \approx \frac{\mathcal{B}}{(\beta-1)(\beta-2)} \frac{1}{W^{\beta-2}}. \tag{48}$$

Introducing it in Eq. (46) and using a simple change of variable

$$\left[\frac{n_0 t \mathcal{B}}{(\beta-1)(\beta-2)} \right]^{-1/(\beta-2)} V = z, \tag{49}$$

one obtains the scaling form (2) with F_{II}^β as in Eq. (15) and the scaling parameters (13).

Class III, Eq. (18). Finally, for distributions with finite support $\phi(V)$ one has

$$\alpha^{as}(W) \approx \frac{\mathcal{C}}{(\beta+1)(\beta+2)} (V_c - W)^{\beta+2}, \quad W \leq V_c. \tag{50}$$

The change of variable

$$\left[\frac{n_0 t \mathcal{C}}{(\beta+1)(\beta+2)} \right]^{1/(\beta+2)} (V_c - W) = -z \tag{51}$$

leads to the scaling form (2) with F_{III}^β given by Eq. (22) and the scaling parameters in Eq. (21).

C. Mean velocity of the leader

Using Eq. (43) and the properties (B3)–(B6) of $\alpha(V)$, followed by a double integration by parts, the mean velocity of the leader can be written as

$$\langle V(t) \rangle = \int_0^\infty dV V P(V,t)$$

$$= \mu e^{-n_0 t \mu} + n_0 t \int_0^\infty dV \left[\alpha(V) - V \frac{d\alpha(V)}{dV} \right] e^{-n_0 t \alpha(V)}, \tag{52}$$

or, with a change of variable

$$Z = n_0 t \alpha(V) \tag{53}$$

the above integral becomes

$$\langle V(t) \rangle = \mu e^{-n_0 t \mu} + \int_0^{n_0 t \mu} dZ \left(\frac{\alpha(V)}{|d\alpha(V)/dV|} + V \right) e^{-Z}. \tag{54}$$

The quantity $\left(\frac{\alpha(V)}{|d\alpha(V)/dV|} + V \right)$ has to be expressed as a function of Z using Eq. (53).

In the *long-time limit* of $t \gg (n_0 \mu)^{-1}$, the first term on the rhs of Eq. (54) becomes negligible, and the dominant contribution to the integral in the second term comes only from the large-V sector. Therefore

$$\langle V(t) \rangle \approx n_0 t \int_0^\infty dZ \left(\frac{\alpha^{as}(V)}{|d\alpha^{as}(V)/dV|} + V \right) e^{-Z}, \tag{55}$$

with $Z = n_0 t \alpha^{as}(V)$, so the long-time behavior of the mean velocity is determined by the tail of the parent distribution $\phi(V)$. Therefore its expression is the same for all the distribution $\phi(V)$ belonging to one of the three classes described above, and is given, respectively, by Eqs. (9), (17), and (24) above.

Note that these long-time expressions of the mean leader velocity can also be obtained directly by using the specific scaling forms (2) of $P(V,t)$.

IV. COLLISION STATISTICS

A. Growing regime

We are interested in the statistics of the number of collisions n_c that the leader undergoes until time t in the growing regime. As before, we then fix the time t and take first the thermodynamic limit $N \to \infty$, $L \to \infty$ keeping the density n_0 fixed. If V is the leader's velocity at time t, then the probability that it undergoes one collision in the small interval dt is $n_0 \alpha(V) dt$, see Sec. III A. Taking the average over the distribution $P(V,t)$ of V, one obtains

$$\frac{d\langle n_c(t)\rangle}{dt} = n_0 \int_0^\infty dV \alpha(V) P(V,t) = n_0 \langle \alpha(V)\rangle. \quad (56)$$

The long-time behavior is obtained by considering in the above equation the asymptotic form (2) of $P(V,t)$. One notices that the most important contribution to the integral over V comes from the region $V \approx a_i(t)$, $i = \mathrm{I,II,III}$. For the three classes of parent distributions $\phi(V)$ one obtains, respectively,

Class I.

$$\frac{d\langle n_c(t)\rangle}{dt} \approx n_0 \int_0^\infty dV \frac{\mathcal{A}}{\delta^2} V^{\beta+2-2\delta} e^{-V^\delta} \frac{1}{b_I(t)} F_I\left(\frac{V-a_I(t)}{b_I(t)}\right)$$

$$\approx \frac{1}{t\delta^2} \int_0^\infty dV \exp[-V^\delta + (\beta+2-2\delta)\ln V$$

$$+ \ln(n_0 t \mathcal{A})] \frac{1}{b_I(t)} F_I\left(\frac{V-a_I(t)}{b_I(t)}\right)$$

$$\approx \frac{1}{t} \int_{-\infty}^\infty dz e^{-z} F_I(z) \approx \frac{1}{2t}. \quad (57)$$

Class II.

$$\frac{d\langle n_c(t)\rangle}{dt} \approx \frac{1}{t} \int_0^\infty dz z^{2-\beta} F_{II}^\beta(z) \approx \left(\frac{\beta-1}{2\beta-3}\right) \frac{1}{t}. \quad (58)$$

Class III.

$$\frac{d\langle n_c(t)\rangle}{dt} \approx \frac{1}{t} \int_{-\infty}^0 dz(-z)^{\beta+2} F_{III}^\beta(z) \approx \left(\frac{\beta+1}{2\beta+3}\right) \frac{1}{t}. \quad (59)$$

These results lead obviously to the $\propto \ln(n_0 t)$ asymptotic growth of $\langle n_c(t)\rangle$, as described by Eqs. (29) and (30).

The mean square number of collisions until time t is given by the following integral expression:

$$\langle n_c^2(t)\rangle = 2 \int_0^t dt_1 \int_{t_1}^t dt_2 \int_0^\infty dV_1 \int_{V_1}^\infty dV_2 \int_{V_1}^{V_2} dU$$

$$\times n_0(U-V_1)\phi(U)P(V_2,t_2|U,t_1)n_0\alpha(V_2). \quad (60)$$

The kernel of this fivefold integral corresponds to (i) having one collision in the interval dt_1 around t_1, provided that the leader has velocity V_1 at t_1; (ii) as a result of this first collision, an instantaneous change in the leader's velocity from V_1 to $U > V_1$; and (iii) a second collision in the interval dt_2 around t_2, provided that the leader has velocity $V_2 \geqslant U$ at t_2 ($t_2 > t_1$) and that it had the velocity U at t_1^+. The corresponding conditional probability density $P(V_2,t_2|U,t_1)$ can be eas-

ily computed using Eq. (40) and integrating over the spatial coordinates, and it is found to be

$$P(V_2,t_2|U,t_1) = e^{-n_0(t_2-t_1)\alpha(U)}\delta(V_2-U)$$

$$+ n_0(t_2-t_1)\phi(V_2)\int_U^{V_2} dW e^{-n_0(t_2-t_1)\alpha(W)}$$

$$\times \theta(V_2-U). \quad (61)$$

Finally, one has to average over all the realizations of the leader's trajectory that fulfills conditions (i)–(iii).

From Eqs. (60), (61), (29), and (30), one can infer the long-time behavior $\propto \ln(n_0 t)$ of the variance of n_c, as resumed in Eqs. (31) and (32). This calculation is rather tedious and we have carried it out explicitly only in a special case when $\phi(V) = \exp(-V)$ (see Sec. V). However, the results for other cases can be inferred by matching the late-time growing regime results with that of the stationary regime results derived in Ref. [21] at the crossover time $t = t^*(N)$. This is explained in detail in Sec. IV B.

B. Stationary regime and the crossover time

As already mentioned in the Introduction, the Jepsen gas for a finite N is completely equivalent to the model of an evolutionary dynamics for a quasispecies model introduced in [19,20]. The collision statistics of the leader in the stationary regime has been studied before both numerically [19,20] and analytically [21], and the results in Eqs. (25) and (27) were derived. Here we will derive these results through a different method that will also allow us to estimate the crossover time $t^*(N)$ for the three classes of $\phi(V)$.

Recall that the crossover time $t^*(N)$ is the time at which the leader aquires its final, maximum velocity. So, at $t = t^*(N)$ the typical time-dependent velocity of the leader in the growing regime matches with the typical value of $V_{\max} = \max(V_0, V_1, \ldots, V_N)$. To estimate $V_{\max,\mathrm{typ}}$, we recall that the V_i's are i.i.d random variables each drawn from $\phi(V)$, and therefore for $N \gg 1$:

$$\mathrm{Prob}(V_{\max} \leqslant \mathcal{V}) = \prod_{i=0}^N \mathrm{Prob}(V_i \leqslant \mathcal{V}) = \left[\int_0^{\mathcal{V}} dU \phi(U)\right]^{N+1}$$

$$= \left[1 - \int_{\mathcal{V}}^\infty dU \phi(U)\right]^{N+1}$$

$$\approx \exp\left[-N \int_{\mathcal{V}}^\infty dU \phi(U)\right]$$

$$= \exp[N\alpha'(\mathcal{V})], \quad (62)$$

where the function $\alpha(z)$ is defined in Eq. (39).

Class I. In this case, Eq. (62) generates the cumulative of the Fisher-Tippett-Gumbel PDF,

$$\mathrm{Prob}(V_{\max} \leqslant \mathcal{V}) \approx \exp\left[-\exp\left(-\frac{\mathcal{V}-\tilde{a}_I(N)}{\tilde{b}_I(N)}\right)\right], \quad (63)$$

where $\tilde{a}_I(N)$ and $\tilde{b}_I(N)$ are given in Eq. (4). So in this case the typical value of $V_{\max}$ is $V_{\max,\mathrm{typ}} \approx \tilde{a}_I(N)$. On the other

hand, as indicated by Eqs. (2), (6), and (9), the typical leader velocity at time t is roughly $V_{\text{typ}}(t) \approx a_I(t)$. Matching $V_{\text{max,typ}} = V_{\text{typ}}[t^*(N)]$ gives the crossover time

$$t^*(N) \approx \frac{\delta}{n_0} N. \tag{64}$$

Class II. In this case, one obtains the cumulative of the Fréchet PDF,

$$\text{Prob}(V_{\text{max}} \leq \mathcal{V}) \approx \exp\left[-\frac{N\mathcal{B}}{\beta - 1} \mathcal{V}^{1-\beta} \right], \tag{65}$$

and hence

$$V_{\text{max,typ}} \approx N^{1/(\beta-1)}. \tag{66}$$

On the other hand, according to Eqs. (2), (13), and (17), the typical leader velocity

$$V_{\text{typ}}(t) \approx \left[\frac{\mathcal{B} n_0 t}{(\beta - 1)(\beta - 2)} \right]^{1/(\beta-2)}. \tag{67}$$

Then the crossover time is estimated as

$$t^*(N) \approx N^{(\beta-2)/(\beta-1)}. \tag{68}$$

Class III. Finally, one has in this case the cumulative of the Weibull PDF,

$$\text{Prob}(V_{\text{max}} \leq \mathcal{V}) \approx \exp\left[-\frac{N\mathcal{C}}{\beta + 1} (V_c - \mathcal{V})^{\beta+1} \right], \tag{69}$$

indicating

$$V_c - V_{\text{max,typ}} \approx N^{-1/(\beta+1)}. \tag{70}$$

The typical leader velocity is given, through Eqs. (2), (21), and (24), by

$$V_c - V_{\text{typ}}(t) \approx \left[\frac{\mathcal{C} n_0 t}{(\beta + 1)(\beta + 2)} \right]^{-1/(\beta+2)}, \tag{71}$$

and thus the crossover time is

$$t^*(N) \approx N^{(\beta+2)/(\beta+1)}. \tag{72}$$

Substituting Eqs. (64), (68), and (72) in Eq. (29), one can then compute the saturation value of the mean collision number $\langle n_c^*(N) \rangle = \langle n_c[t=t^*(N)] \rangle$ as a function of N as stated in Eq. (25). We thus recover, through this alternative method, the results derived in Ref. [21].

Concerning the variance, the results in Eqs. (27) and (28) for the stationary regime were derived exactly in Ref. [21]. In order that the variance in the growing regime matches with that in the stationary regime at the crossover time $t=t^*(N)$, it follows immediately that in the growing regime the variance at late times must behave as in Eqs. (31) and (32). We have verified this prediction by direct calculation for the special case $\phi(V) = \exp(-V)$ (see the next section). However, although desirable, a direct calculation of the variance in the growing regime for the other cases, using the method outlined in the previous section, seems too tedious.

V. EXACT RESULTS FOR ALL TIME IN A SPECIAL CASE

A particular example that can be studied in full analytical detail for all t is

$$\phi(V) = e^{-V} \quad (V \geq 0), \tag{73}$$

which pertains to Class I with $\delta = 1$. By straightforward calculations one obtains from Eqs. (44), (52), and (56), respectively, the following.

(i) The probability distribution function for the velocity of the leader

$$P(V,t) = e^{-(n_0 t + V)} + n_0 t e^{-V} [\text{Ei}(-n_0 t) - \text{Ei}(-n_0 t e^{-V})], \tag{74}$$

with the scaling form (2) corresponding to the function (7) with the parameters

$$a_I(t) = \ln(n_0 t), \quad b_I = 1. \tag{75}$$

(ii) The mean velocity is therefore

$$\langle V(t) \rangle = \ln(n_0 t) + 1 + C - \text{Ei}(-n_0 t), \tag{76}$$

which for long times is dominated by the logarithmic term.

(iii) The mean value of the number of collisions the leader undergoes until time t:

$$\langle n_c(t) \rangle = \frac{1}{2} [\ln(n_0 t) + C - \text{Ei}(-n_0 t)], \tag{77}$$

with the logarithmic asymptotic increase.

(iv) Finally, the calculation of the variance of the number of collisions is rather lengthy (see Appendix D), but its long-time behavior is simply given by

$$\langle n_c^2(t) \rangle - \langle n_c(t) \rangle^2 \approx \frac{1}{4} \ln(n_0 t), \tag{78}$$

in agreement with Eqs. (31) and (32).

VI. CONCLUSIONS

In this paper we have studied the extremal dynamics in a one-dimensional Jepsen gas of $(N+1)$ particles, initially confined in a box $[-L, 0]$ with uniform density and with each particle having an independently distributed initial positive velocity drawn from an arbitrary distribution $\phi(V)$. We have computed analytically the velocity distribution of the leader (or the rightmost particle) at time t, and also the mean and the variance of the number of collisions undergone by the leader up to time t. We have shown that for a given $N \gg 1$, there is a crossover time $t^*(N)$ that separates a stationary regime $[t > t^*(N)]$ from a growing regime $[1 \ll t \ll t^*(N)]$. While in the stationary regime, the leader velocity becomes time-independent and follows the standard extremal laws of i.i.d random variables, it has an entirely different universal scaling behavior in the growing regime. The associated scaling functions in the growing regime belong to three different universality classes depending only on the tail of $\phi(V)$, and they have been computed explicitly in Eqs. (7), (15), and (22). These dynamical extremal scaling functions are mani-

festly different from the standard EVS scaling functions of i.i.d random variables.

Similarly, we have shown that in the growing regime, the mean and the variance of the number of collisions of the leader up to time t increases logarithmically with t, with universal prefactors that were computed explicitly in Eqs. (30) and (32). Also, as a by-product, we have provided an alternate derivation of the stationary regime results for the collision statistics (the mean and the variance) of the leader that were obtained in [21] through a completely different approach. While in this paper we were able to compute only the mean and the variance of the number of collisions, it would be interesting to obtain the full distribution of the number of collisions up to time t, and to compare it with the previously incorrectly suggested Poisson distribution [33], which remains a challenging open problem.

We have computed here the velocity distribution and the statistics of the number of collisions *separately* in the growing regime and in the stationary regime. It would be interesting to obtain the exact crossover functions that interpolate between the two regimes. For example, for the velocity distribution of the leader this crossover scaling function can, in principle, be computed from our general result in Eq. (36), which is valid for all times t and all values of N.

The model studied here can also be considered as a simple toy model of biological evolution [19–21], on which some results, numerical and analytical, were known before but they were mostly restricted to the stationary regime [19–21]. For example, the number of overtaking events of the leader, i.e., the number of punctuation events until the emergence of the best fitted species were studied before [19–21]. However, the authors in Ref. [20] also studied analytically the distribution of the "label" of the fittest species in the growing regime, and by matching the typical leader's label at time t

with that of the final leader, they were able to extract the crossover time $t^*(N)$. In this paper, we have studied a complementary quantity in the growing regime, namely the distribution of the "fitness" (velocity) of the fittest species. The crossover times $t^*(N)$ extracted from both of these distributions are in agreement with each other. The method presented in this paper may also be useful to study the dynamics of other interesting observables in the context of biological evolution, such as, for example, the persistence of the leader genotype, and the distribution of the time interval between two successive punctuation events.

APPENDIX A: DERIVATION OF EQ. (36) AND THE NORMALIZATION OF $P_L(X,V,t|V_0)$

We present below the main steps of the derivation of Eq. (36), see also [32]. In order to pursue the calculations, it is convenient to use the following integral representation of the Kronecker delta:

$$\delta_{Kr}(p,q) = \oint_\Gamma \frac{dz}{2\pi i z} z^{p-q}, \tag{A1}$$

where p,q are integers, and Γ is the unit circle centered at the origin in the complex z plane. Using this representation in Eq. (34), it results in

$$
\begin{aligned}
P_L(X,V,t|V_0) &= \left\langle \sum_{p=0}^{N} \delta(X-X_p-V_p t)\delta(V-V_p) \oint_\Gamma \frac{dz}{2\pi i z} z^N \times \prod_{j=0,j\neq p}^{N} z^{-\theta(X_p+V_p t - X_j - V_j t)} \right\rangle \\
&= \oint_\Gamma \frac{dz}{2\pi i z} z^N \Biggl\{ \delta(X-V_0 t)\delta(V-V_0)\left\langle \prod_{j=1}^{N} z^{-\theta(V_0 t - X_j - V_j t)} \right\rangle \\
&\quad + \left\langle \sum_{p=1}^{N} \delta(X-X_p-V_p t)\delta(V-V_p) z^{-\theta(X_p+V_p t - V_0 t)} \prod_{j=1,j\neq p}^{N} z^{-\theta(X_p+V_p t - X_j - V_j t)} \right\rangle \Biggr\}.
\end{aligned}
\tag{A2}
$$

Here $\theta(\cdots)$ is the Heaviside unit step function. Due to the initial statistical independence of the particles, one has

$$\left\langle \prod_{j=1}^{N} z^{-\theta(V_0 t - X_j - V_j t)} \right\rangle = \langle z^{-\theta(V_0 t - X_j - V_j t)} \rangle^N = \langle z^{-\theta(X - X_j - V_j t)} \rangle^N, \tag{A3}$$

where the last equality results because of the factor $\delta(X-V_0 t)$ in the corresponding term of Eq. (A2). Also

$$
\begin{aligned}
&\left\langle \sum_{p=1}^{N} \delta(X-X_p-V_p t)\delta(V-V_p) z^{-\theta(X_p+V_p t - V_0 t)} \prod_{j=1,j\neq p}^{N} z^{-\theta(X_p+V_p t - X_j - V_j t)} \right\rangle \\
&\quad = N \langle \delta(X-X_p-V_p t)\delta(V-V_p) \rangle z^{-\theta(X-V_0 t)} \times \langle z^{-\theta(X-X_j-V_j t)} \rangle^{N-1}.
\end{aligned}
\tag{A4}
$$

One has to compute then

$$\langle z^{-\theta(X-X_j-V_jt)} \rangle = \frac{1}{L}\int_{-L}^{0} dX_j \int_{0}^{\infty} dV_j \phi(V_j) z^{-\theta(X-X_j-V_jt)} = \frac{1}{L}\int_{-L}^{0} dX_j \left[z^{-1}\int_{0}^{(X-X_j)/t} dV_j \phi(V_j) + \int_{(X-X_j)/t}^{\infty} dV_j \phi(V_j) \right]$$

$$= z^{-1} + (1-z^{-1})\frac{1}{L}\int_{-L}^{0} dX_j \int_{(X-X_j)/t}^{\infty} dV_j \phi(V_j) = z^{-1} + (1-z^{-1})\left[\int_{(X+L)/t}^{\infty} dV_j \phi(V_j) + \frac{1}{L}\int_{X/t}^{(X+L)/t} dV_j \phi(V_j)(V_jt-X) \right]$$

$$\equiv z^{-1}A(z,X/t|L/t), \tag{A5}$$

where $A(z,X/t|L/t)$ is given by Eq. (37). Note also that

$$\langle \delta(X-X_p-V_pt)\delta(V-V_p) \rangle = \frac{1}{L}\int_{-L}^{0} dX_p \int_{-\infty}^{\infty} dV_p \phi(V_p)\delta(X-X_p-V_pt)\delta(V-V_p) = \frac{1}{L}\int_{-L}^{0} dX_p \delta(X-X_p-Vt)\phi(V)$$

$$= \frac{1}{L}\phi(V)\theta(Vt-X)\theta(X-Vt+L), \tag{A6}$$

and also

$$z^{-\theta(X-V_0t)} = z^{-1} + (1-z^{-1})\theta(V_0t-X)$$
$$= z^{-1}[1+(z-1)\theta(V_0t-X)]. \tag{A7}$$

Finally, replacing the results (A3)–(A7) in Eq. (A2), one obtains the expression (36) of the conditional probability density for the leader's coordinate and velocity. One can also check the normalization of this one. Indeed, consider

$$\mathcal{N} = \int_{-\infty}^{\infty} dX \int_{-\infty}^{\infty} dV P_L(X,V,t|X_0=0,V_0)$$

$$= \oint_{\Gamma} \frac{dz}{2\pi i z}\left\{ [A(z,V_0|L/t)]^N + \frac{N}{L}\int_{-\infty}^{\infty} dX \int_{X/t}^{(X+L)/t} dV\phi(V)\right.$$

$$\left. \times [A(z,X/t|L/t)]^{N-1}[1+(z-1)\theta(V_0t-X)]\right\}$$

$$= [A(0,V_0|L/t)]^N + \frac{N}{L}\int_{-\infty}^{\infty} dX \int_{X/t}^{(X+L)/t} dV\phi(V)$$

$$\times [A(0,X/t|L/t)]^{N-1}\theta(X-V_0t), \tag{A8}$$

where for the last equality we used the theorem of residues. But

$$\frac{\partial}{\partial X}[A(0,V_0|L/t)]^N = \frac{N}{L}[A(0,X/t|L/t)]^{N-1}\int_{X/t}^{(X+L)/t} dU\phi(U), \tag{A9}$$

and therefore

$$\mathcal{N} = [A(0,V_0|L/t)]^N + \int_{-\infty}^{\infty} dX\theta(X-V_0t)\frac{\partial}{\partial X}[A(0,X/t|L/t)]^N$$

$$= [A(0,V_0|L/t)]^N + \int_{V_0t}^{\infty} dX\frac{\partial}{\partial X}[A(0,X/t|L/t)]^N$$

$$= \lim_{X\to\infty}[A(0,X/t|L/t)]^N = 1, \tag{A10}$$

which proves the normalization of $P_L(X,V,t|V_0)$.

APPENDIX B: PROPERTIES OF THE FUNCTION $\alpha(W)$

Starting from its definition (39), one can easily obtain the following properties of the function $\alpha(W)$:

(i) $\alpha(W)$ is a strictly decreasing function of W and

$$\text{(ii)} \quad \alpha(W) = \mu - W \quad \text{for } W \leq 0. \tag{B1}$$

In particular, $\alpha(0) = \mu$, the first moment of $\phi(V)$.
$$\tag{B2}$$

$$\text{(iii)} \quad \alpha(W) \to 0 \quad \text{for } W \to \infty. \tag{B3}$$

$$\text{(iv)} \quad \alpha(W) \to -\infty \quad \text{for } W \to -\infty. \tag{B4}$$

$$\text{(v)} \quad \frac{d\alpha(W)}{dW} = -\int_{W}^{\infty} dU\phi(U). \tag{}$$

$$\text{Note} \quad \left.\frac{d\alpha(W)}{dW}\right|_{W=0} = -1. \tag{B5}$$

$$\text{(vi)} \quad \frac{d^2\alpha(W)}{dW^2} = \phi(W). \tag{B6}$$

Using these relations, one can easily obtain Eq. (44) from Eq. (43).

APPENDIX C: THE SCALING BEHAVIOR OF $P(V,t)$ FOR THE CLASS I OF PARENT DISTRIBUTIONS $\phi(V)$

With the expression (47) of α^{as}, one finds from Eq. (46)

$$P(V,t) \approx n_0 t \mathcal{A} V^\beta e^{-V^\delta}\int_{0}^{V} dW e^{-e^{S(W)}}, \tag{C1}$$

where

$$S(W) = W^\delta - \ln(n_0 t \mathcal{A}/\delta^2) - (\beta+2-2\delta)\ln W. \tag{C2}$$

Consider

$$W = a_I(t) + y, \qquad \text{(C3)}$$

where y is small compared to $a_I(t)$. Then

$$S[a_I(t) + y] = [a_I(t)]^\delta - \ln(n_0 t \mathcal{A}/\delta^2) - (\beta + 2 - 2\delta)\ln a_I(t)$$
$$+ \left[\frac{\delta}{[a_I(t)]^{1-\delta}} - \frac{(\beta + 2 - 2\delta)}{a_I(t)} \right] y + O(y^2). \qquad \text{(C4)}$$

Set

$$[a_I(t)]^\delta - \ln(n_0 t \mathcal{A}/\delta^2) - (\beta + 2 - 2\delta)\ln a_I(t) = 0, \quad \text{(C5)}$$

so, to leading order for large t:

$$[a_I(t)]^\delta \approx \ln(n_0 t \mathcal{A}/\delta^2) + (\beta + 2 - 2\delta)/\delta \ln \ln(n_0 t \mathcal{A}/\delta^2), \qquad \text{(C6)}$$

and thus

$$a_I(t) \approx [\ln(n_0 t \mathcal{A}/\delta^2)]^{1/\delta}. \qquad \text{(C7)}$$

Then

$$S[W = a_I(t) + y] = \frac{W - a_I(t)}{b_I(t)}, \qquad \text{(C8)}$$

where

$$b_I(t) \approx \frac{1}{\delta}[\ln(n t \mathcal{A}/\delta^2)]^{(1-\delta)/\delta}. \qquad \text{(C9)}$$

Now, from Eq. (C1),

$$P(V, t) \approx n_0 t b_I(t) \mathcal{A} V^\beta e^{-V^\delta} \int_0^V dW e^{-[W - a_I(t)]/b_I(t)}$$
$$\approx b_I(t) \exp[-V^\delta + \ln(n_0 t \mathcal{A}) + \beta \ln V] \int_{-\infty}^z dU e^{-e^{-U}}, \qquad \text{(C10)}$$

where $z = [V - a_I(t)]/b_I(t)$. One has

$$\exp[-V^\delta + \ln(n_0 t \mathcal{A}) + \beta \ln V] \approx \frac{1}{b_I^2(t)} e^{-z}, \qquad \text{(C11)}$$

and thus obtains finally Eq. (2) for case I.

APPENDIX D: THE MEAN SQUARE DEVIATION OF $n_c(t)$ FOR $\phi(V) = \exp(-V)$

Using Eqs. (60) for computing $\langle n_c^2(t) \rangle$, for the particular case of $\phi(V) = \exp(-V)(V \geq 0)$ one has $\alpha(V) = \exp(-V)$, $P(V, t)$ given by Eq. (74), and the conditional velocity distribution function (61)

$$P(V_2, t_2 | U, t_1) = e^{-n_0(t_2 - t_1)e^{-V_2}} \delta(V_2 - U) + n_0(t_2 - t_1)e^{-V_2}$$
$$\times [\mathrm{Ei}(-n_0(t_2 - t_1)e^{-U}) - \mathrm{Ei}(-n_0(t_2 - t_1)e^{-V_2})]$$
$$\times \theta(V_2 - U). \qquad \text{(D1)}$$

Then, after lengthy calculations, Eq. (60) leads to

$$\langle n_c^2(t) \rangle = \frac{1}{8}\ln^2(n_0 t) + \left(\frac{C}{2} + \frac{1}{4} \right)\ln(n_0 t) - \frac{3}{2n_0 t} - 2Ce^{-n_0 t}$$
$$- \frac{e^{-n_0 t}}{2n_0 t} + \left(\frac{7}{4} - \frac{C}{2} \right)\mathrm{Ei}(-n_0 t) - \ln(n_0 t)\mathrm{Ei}(-n_0 t)$$
$$+ 2e^{-n_0 t}\,\mathrm{Ei}(-n_0 t) + \left(\frac{9C}{4} + \frac{C^2}{2} + \frac{\pi^2}{12} \right) + J, \quad \text{(D2)}$$

where

$$J = \frac{1}{2}\mathcal{P} \int_0^{n_0 t} \frac{d\tau}{\tau}[e^{-n_0 \tau}\mathrm{Ei}(-n_0(t - \tau)) + \mathrm{Ei}(-n_0 \tau)]. \quad \text{(D3)}$$

Here $\mathcal{P}$ designates the principal part of the above integral.

In the long-time limit $n_0 t \gg 1$, the main contribution to the value of $\langle n_c^2(t) \rangle$ comes from the logarithmic terms,

$$\langle n_c^2(t) \rangle \approx \frac{1}{8}\ln^2(n_0 t) + \left(\frac{C}{2} + \frac{1}{4} \right)\ln(n_0 t). \qquad \text{(D4)}$$

Combining this result with the expression (77) of $\langle n_c(t) \rangle$, one obtains finally the asymptotic result (78).

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